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A Case study on simulation of heat equation by Crank-Nicolson Method in Accordance with digital image processing

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Abstract— We present an approach for computing the temporal variation of heat destitution for a (2+1) dimensional heat equation of some originally heated coins without the use of an external source in this paper. The algorithm is based on digital image processing, in which an image is turned into an array of small integers, known as pixels, which represent a physical quantity, which in our case is temperature. Our goal is to provide the initial temperature of the image of our model physical domain, which will be used to determine the heat equation's initial properties. Finally, using the Crank-Nicholson Method, solve the model equation.

Index Terms— Heat equation, Crank-Nicolson Method, Digital Image Processing, Discretization, MATLAB.

1 Introduction

The heat equation is a parabolic partial differential equation that represents the temperature variation in a given area over time [1]. The heat equation is used to solve problems arising in the fields of engineering, fluid mechanics, atmospheric science, climate physics, weather forecasting, and solar physics. The term "temperature" refers to a physical number that expresses how hot or cold something is. It's a type of heat energy [2]. The earth's temperature is steadily rising. In this regard, the heat equation, as well as the wave equation, plays a significant role in expressing the earth's worming phenomena and some of its implications. [3]. There are many numerical methods to solve this equation such as finite difference method, Crank-Nicolson method, Leapfrog method etc. The Crank Nicolson technique has a lot of support in the financial literature, and it appears to be the de-facto finite difference strategy for one-factor and two-factor Black Scholes equations [4]. This methodology provides second-order precision and is numerically stable. To solve the heat equation, we mostly employ the Crank-Nicolson method. We will solve for the heat equation value at time T by applying the heat equation to a picture with the initial condition that the temperature distribution equals the image intensity. Image processing, as we all know, is a technique for executing operations on a photograph in order to improve it or extract relevant data. [5]. It's a type of signal processing in which an image serves as the input and the output is either an image or the image's characteristics/features. Digital image processing techniques facilitate the use of computers to edit digital photographs. We used an image of a coin in this study and then employed this method to show the various temperatures.

2. Mathematical background

2.1 Literature view

The Crank Nicolson method is the most commonly used method for solving parabolic partial differential equations. The Crank Nicolson Method for solving heat equations was developed by John Crank and Phyllis Nicholson in the mid-twentieth century [6]. It was looked into a method for numerically evaluating heat conduction partial differential equations. This combines the accuracy of a second-order approach in both space and time with the stability of an implicit method. First-order temporal truncation issues exist in both the explicit (ahead Euler) and implicit (reverse Euler) approaches [7]. This means that short time increments must be employed to achieve an accurate result. Furthermore, the explicit technique has additional limits due to its stability

Due to concerns about computer efficiency, excessively small time steps are unaffordable in many practical applications. It's a higher-order implicit method (in time). The Crank-Nicolson technique is numerically stable at each time level and only requires the solution of a basic system of linear equations (specifically, a tridiagonal system) [8].

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2.2 Discretization of two dimensional heat equations by Crank-Nicholson Method:

Consider a 2D heat equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

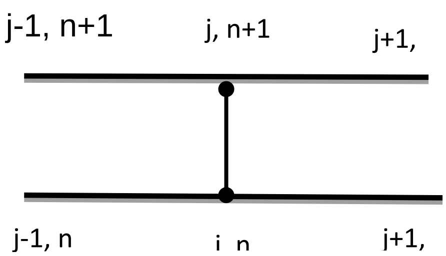

Figure 1: Fictitious diagram of Crank-Nicholson Method

Here,

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} \ldots\ldots\ldots\ldots\ldots (2)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} \ldots\ldots\ldots\ldots\ldots (3)$$

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Now putting $n+1/2$ and n, respectively for $\frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x^2}$

Now from the heat equation we get

$$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^n}{\frac{\Delta t}{2}} = k\left[\frac{\partial^2 u}{\partial x^2}\Big|_n + \frac{\partial^2 u}{\partial y^2}\Big|_{n+\frac{1}{2}}\right]$$

Here taking

$$\Delta x = \Delta y \quad , \quad r = \frac{\Delta t}{(\Delta x)^2}$$

$$u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^n = \frac{r}{2}\left[u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n + u_{i,j+1}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i,j-1}^{n+\frac{1}{2}}\right]$$

$$u_{i,j}^{n+\frac{1}{2}} = u_{i,j}^n + \frac{r}{2}\left[u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n + u_{i,j+1}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i,j-1}^{n+\frac{1}{2}}\right] \ldots (5)$$

Again putting $n+1$ and $n+1/2$, respectively for $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{(\Delta x)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i,j-1}^{n+\frac{1}{2}}}{(\Delta y)^2}$$

$$\frac{\partial u}{\partial t} = \frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}}}{\frac{\Delta t}{2}}$$

Here taking

$$\Delta x = \Delta y \quad , \quad r = \frac{\Delta t}{(\Delta x)^2}$$

$$u_{i,j}^{n+1} = u_{i,j}^{n+\frac{1}{2}} + \frac{r}{2}\left[u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1} + u_{i,j+1}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i,j-1}^{n+\frac{1}{2}}\right] \ldots (6)$$

From equation (5) and (6) we get

$$u_{i,j}^{n+1} = u_{i,j}^n + \frac{r}{2}\left[u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n + u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}\right]$$

$$\Rightarrow 2(u_{i,j+1} - u_{i,j}) = r\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right]$$

$$\Rightarrow 2u_{i,j+1} - 2u_{i,j} = r\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right]$$

$$\therefore (2+2r)u_{i,j+1} - ru_{i-1,j+1} - ru_{i+1,j+1} = (2-2r)u_{i,j} + ru_{i+1,j} + ru_{i-1,j} \ldots (B)$$

The equation (B) is called the Crank Nicholson Method which is accurate in both time and space. This equation can be written in the matrix form as

$$AU^{j+1} = BU^j \; for \; j = 0,1,2,3 \ldots\ldots\ldots$$

Where,

$$A = \begin{bmatrix} 2+2r & -r & 0 & & 0 & 0 & 0 \\ -r & 2+2r & -r & \ldots & 0 & 0 & 0 \\ 0 & -r & 2+2r & & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 2+2r & -r & 0 \\ 0 & 0 & 0 & & -r & 2+2r & -r \\ 0 & 0 & 0 & \ldots & 0 & -r & 2+2r \end{bmatrix}$$

$$B = \begin{bmatrix} 2-2r & r & 0 & & 0 & 0 & 0 \\ r & 2-2r & r & \cdots & 0 & 0 & 0 \\ 0 & r & 2-2r & & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 2-2r & r & 0 \\ 0 & 0 & 0 & & r & 2-2r & r \\ 0 & 0 & 0 & \cdots & 0 & r & 2-2r \end{bmatrix}$$

$$U^j \equiv [u_{1,j} \; u_{2,j} \cdots\cdots\cdots\cdots .. u_{m-1,j}]^t$$

And

$$U^{j+1} \equiv [u_{1,j+1} \; u_{2,j+1} \cdots\cdots\cdots\cdots .. u_{m-1,j+1}]^t$$

3. Numerical Solution of the selected problem:

3.1 Problem statement

Use Crank Nicolson method to solve

$$\frac{\partial^2 u}{\partial x^2} = 16 \frac{\partial u}{\partial t} \quad 0 < x < 1$$

and $t > 0$, given $u(x,0) = 0, u(0,t) = 0$ and $u(1,t) = 100t$. Compute $u(x,t)$ for one time step taking $\Delta x = 1\backslash 4$.

3.2 Solution of the problem

Given, 1D heat equation is

$u_{xx} = 16u_t$

Comparing this equation with $u_{xx} = au_t$

We have $a = 16$.

Given x varies from 0 to 1 with $h = \Delta x = 0.25$

$x = 0, 0.25, 0.50, 0.75, 1$

$i = 0, 1, 2, 3, 4$

Since the value of k is not given

Let $r = 1$, $k = ah^2$

$$\Rightarrow k = (16)\left(\frac{1}{16}\right)$$

$$\Rightarrow k = 1$$

We have to find the value $u(x,t)$ of upto one time step with $k = 1$.

$t = 0,1$

$j = 0,1$

Using the boundary conditions $u(0,t) = 0$

$u_{0,j} = 0$ (First column)

$\therefore u(1,0) = 0, \; u(1,1) = 100(1) = 100$

(Last column)

Given the initial conditions,

$u(x,0) = 0$

$u_{1,0} = 0$

(First row)

The remaining values can be calculated by Crank-Nicholson formula,

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i-1,j+1} + u_{i+1,j+1} \right] \ldots (*)$$

We have the following table

		$x = 0$	$x = 0.25$	$x = 0.50$	$x = 0.75$	$x = 1$
	i \ j	1	2	3	4	5
t = 0	0	0	0	0	0	0
t = 1	1	0	u_1	u_2	u_3	100

Using the formula (*) to the second row we get

$$u_1 = \frac{1}{4}[0 + 0 + 0 + u_2] \;\; \rightarrow u_1 = \frac{1}{4}u_2$$

$$4u_1 - u_2 = 0 \ldots (i)$$

$$u_2 = \frac{1}{4}[0 + 0 + u_1 + u_3] \;\; \rightarrow u_2 = \frac{1}{4}[u_1 + u_3]$$

$$u_1 - 4u_2 + u_3 = 0 \ldots (ii)$$

$$u_3 = \frac{1}{4}[0 + 0 + u_2 + 100] \;\; \rightarrow u_3 = \frac{1}{4}[u_2 + 100]$$

$$u_2 - 4u_3 = -100 \ldots\ldots\ldots\ldots\ldots (iii)$$

Solving equation (i), (ii) and (iii)we get

$u_1 = 1.7857$

$u_2 = 7.1429$

$u_3 = 26.7857$

This is the required solution.

3.3 MATHLAB code:

Let us consider a heat equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

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Where the function $u = u(t,(x,y))$ depends on time and space variables.

MATLAB Procedure

Input:
Show the original image of a coin.

Output:

Step-1
Initial temperature distribution
$\mathrm{x = 1{:}\,size(A,2)}$
$\mathrm{y = 1{:}\,size(A,1)}$

Step-2
Pixel size of A
$\mathrm{[m,n] = size(A)}$

Step-3
Time distribution
$\mathrm{for\ t = 0{:}\,ht{:}{:}\,tfin}$

Step-4
Condition for pixels
$\mathrm{for\ j = 2{:}\,n - 1}$
$\mathrm{for\ i = 2{:}\,m - 1}$

Step-5
Apply the Crank-Nicholson method

$$\mathrm{Anext(i,j) = A(i,j) + r * (A(i,j+1) + A(i+1,j) + A(i-1,j) +}$$

$$\mathrm{A(i-1,j+1) + A(i+1,j+1) - 2 * A(i,j) - 2 * A(i,j+1))}$$

Step-6
Update the temperature value.

Step-7
Compare the final image with the original image

Step-8
Stop.

4. Result and Discussion

We apply our model equation to the pixel grid of our model image of coins to see how the temperature value varies depending on the starting temperature distribution. To determine the profile of temperature distribution after a specific time period, we'll use the Crank-Nicholson Method, which algorithm is described in section 3.3. The heat distribution from the heated coins after 4 minutes from the start of time is shown in Figure (3). Until time 4, we model the evolution. When we compare the finished image to the original image, we can observe that the final image is blurred (see Figure 4).

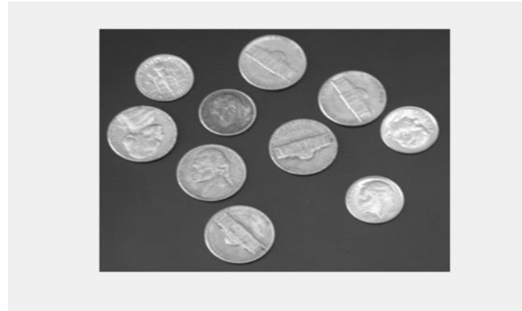

Figure 2: The original image in gray scale

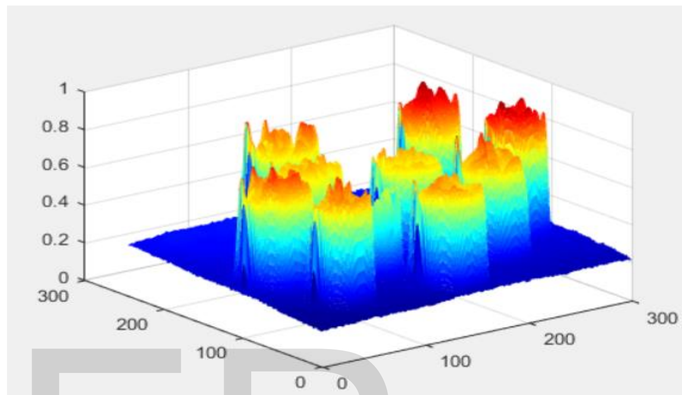

Figure 3: Temperature distribution after 4 minutes by Crank-Nicholson method

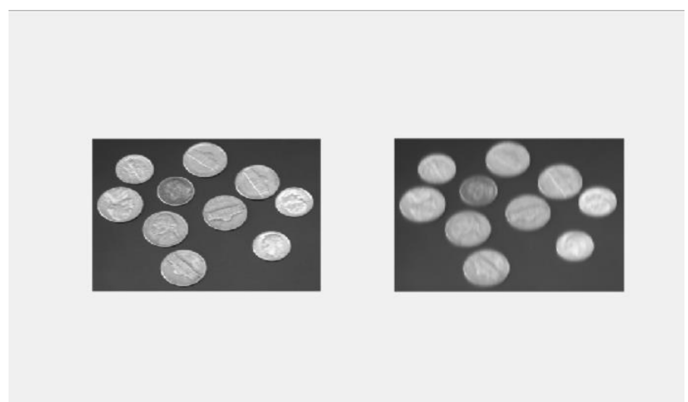

Figure 4: Final comparison image between original image and blurred image by Crank-Nicholson method

Conclusion

The Crank-Nicholson method was used to solve the 2-dimensional heat equation, with the initial conditions and geometric domain being read from a gray image. The heat distribution profile was simulated for distinct points of the domain for 4 minutes from the start and displays those distributions in the above figures, where the color pixels give information about the temperature and the heat distribution profile was simulated for distinct points of the domain for 4 minutes from the start and displays those distributions in the above figures. As a result, it is suggested that digital image processing can be a valuable tool for numerical modeling of the heat equation and other purposes. It should also be emphasized that while a color image may improve the results and applications, our study was based on a gray image for the sake of simplicity and time and space savings.

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